Leopold Mareis (PCIC 24)

Optimizing Experimental Design for Causal Effect Estimation with Partial Measurements

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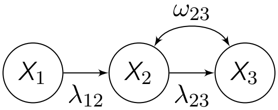
Gaussian Instrumental Variable Model - I



- X_1, X_2, X_3: Instrument, Treatment, Outcome

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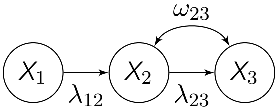
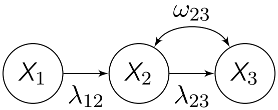


- X_1, X_2, X_3: Instrument, Treatment, Outcome
- Structural Equation Model (SEM)

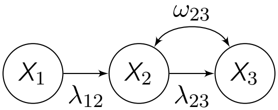
$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ \lambda_{12} \neq 0 & 0 & 0 \\ 0 & \lambda_{23} & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} + \varepsilon, \; \varepsilon \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \omega_{11} & 0 & 0 \\ 0 & \omega_{22} & \omega_{23} \\ 0 & \omega_{23} & \omega_{33} \end{pmatrix} \right)$$

Gaussian Instrumental Variable Model - II



- Full Dataset: $D = (X_1^{(i)}, X_2^{(i)}, X_3^{(i)})_{i \in [n]}$. $\Sigma = Var(X)$. Estimator: $\hat{\lambda}_{23} = \hat{\sigma}_{13} / \hat{\sigma}_{12}$

Gaussian Instrumental Variable Model - II



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- Full and Partial Data: $(X_1^{(j)}, X_2^{(j)}, X_3^{(j)})_{j \in [m_1]}$ for $m_1 c_1$ and $(X_1^{(k)}, X_2^{(k)})_{k \in [m_2]}$ for $m_2 c_2$.
  Estimator:

$$\hat{\lambda}_{23} = \hat{\sigma}_{13}/(\frac{1}{1+\gamma}\hat{\sigma}_{12} + \frac{\gamma}{1+\gamma}\hat{\sigma}_{12}), \quad \gamma = m_2/m_1$$

Gaussian Instrumental Variable Model - II

$$\begin{tikzcd} X_1 \arrow[r, "\lambda_{12}"'] & X_2 \arrow[r, "\lambda_{23}"'] \arrow[r, leftrightarrow, bend left, "\omega_{23}"] & X_3 \end{tikzcd}$$

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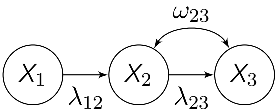
Optimization problem: $\boxed{\min_{m_1, m_2 > 0} Var(\hat{\lambda}_{23}) \quad \text{s.t.} \quad m_1 c_1 + m_2 c_2 \leq b.}$

Asymptotic Variance - Without Budget

Corollary: The asymptotic variance function $v(\gamma)$ *of* $\sqrt{m_1}(\hat{\lambda}_{23} - \lambda_{23})$ *is decreasing for* $\gamma > 0$ *if and only if*

$$\lambda_{12}^2 < -\frac{\omega_{22} + 2\omega_{23}/\lambda_{23}}{\omega_{11}}.$$



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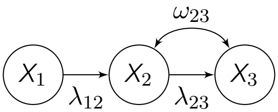

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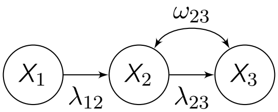
- This can only be the case if $\omega_{23}\lambda_{23} < 0$, so counteracting effect paths.
- A weak instrument scenario ($\lambda_{12}^2\omega_{11}$ small) is necessary when requesting more X_{12} samples.



Asymptotic Variance - With Budget

Theorem: The solution to the optimization problem is given by $(b/c_1, 0)$ *if*

$$\chi_1 := \frac{c_1 - c_2}{c_2}(\sigma_{11}\sigma_{13}^2\sigma_{22} - 2\,\sigma_{11}\sigma_{12}\sigma_{13}\sigma_{23}) \leq 2\,\sigma_{12}^2\sigma_{13}^2 + \sigma_{11}\sigma_{12}^2\sigma_{33} =: \chi_2.$$

Otherwise, a positive amount of X_{12} *samples are requested and the solution reads*

$$\left(\frac{b - \xi c_2 n}{c_1 + \xi c_2}, \frac{\xi(b + c_1 n)}{c_1 + \xi c_2}\right), \text{ where } \xi = \min\left(-1 + \frac{\sqrt{\chi_1\chi_2}}{\chi_2}, \frac{b}{c_2 n}\right).$$

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- Only model parameters and the ratio c_1/c_2 determine whether additional partial samples are requested.

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- Weakening the instrument λ_{12}^2 weakens the condition $\chi_1 < \chi_2$.

Applications

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Real Data:

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Real Data:

- Aggressive Driving: (Lighting, Driving style, Speed). 21% variance reduction. $m^* = (211, 293)$.
- ICU Data: (Caregiver rate, Lasix, Urine). 68% variance reduction. $m^* = (88, 77)$.
- Assembly Line [**?**]: No partial data helpful.

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Thank you for your attention

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Outlook

- Procedure also applicable for general linear SEMs with arbitrary structure under identifiability.
- Gaussianity can be relaxed towards asymptotic normality (via, e.g., the maximum likelihood estimator).