# Semiparametric Inference for Half-Trek Estimators in Linear Structural Equation Models

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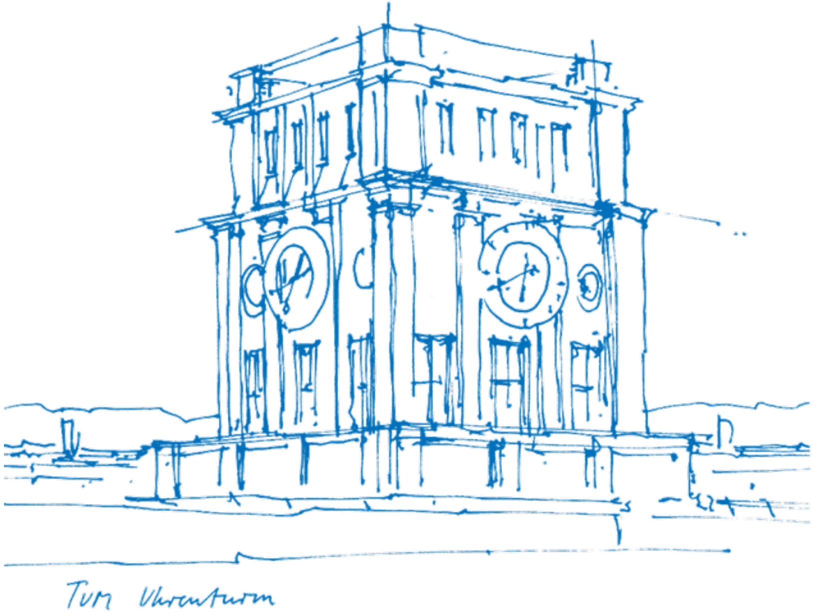
(joint work with Nils Sturma and Mathias Drton)



# A Familiar Summary

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| 1.3 | -1.7 | -0.7 | -0.4 | 1.2 |
| $\vdots$ | | | | |
| 1.5 | -1.7 | 0.2 | 1.6 | -0.5 |

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...

Coefficients:
   Estimate Std. Error t value Pr(>|t|)
X1  1.56692    0.05548  28.245  < 2e-16 ***
X2  0.35090    0.04565   7.687 3.59e-14 ***
X3  0.33046    0.03616   9.139  < 2e-16 ***
X4 -0.54194    0.04816 -11.253  < 2e-16 ***
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Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.959 on 996 degrees of freedom
Multiple R-squared:  0.6833,	Adjusted R-squared:  0.682
F-statistic: 537.3 on 4 and 996 DF,  p-value: < 2.2e-16
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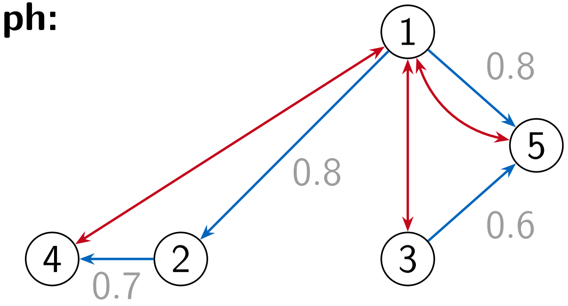


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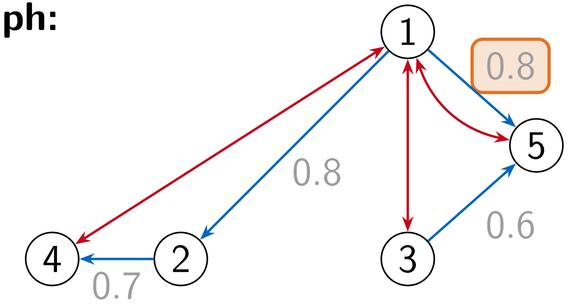


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OLS is *inconsistent* under latent confounding.

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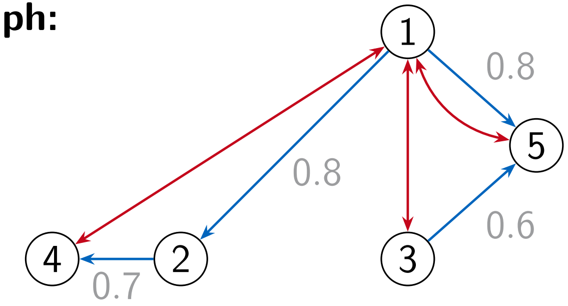


**Our Paper:**

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HTC Estimator Summary
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Node 2  (pa: 1)  [witnesses: 3 (ext)]

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1 -> 2      0.8350      0.0821  10.1770    < 2e-16 ***
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Residual std. dev: 1.0114   Structural R-sq: 0.3879   (n = 1000)
-------------------------------------------------------
Node 4  (pa: 2)  [witnesses: 3 (ext)]
...
-------------------------------------------------------
Node 5  (pa: 1, 3)  [witnesses: 3 (ext), 4 (int)]

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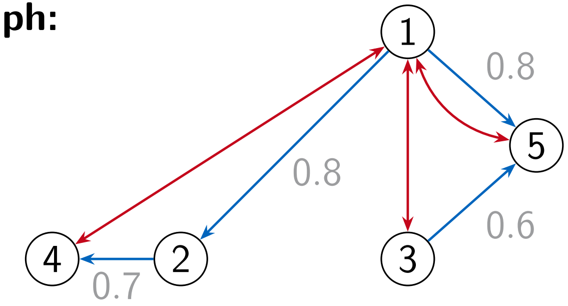


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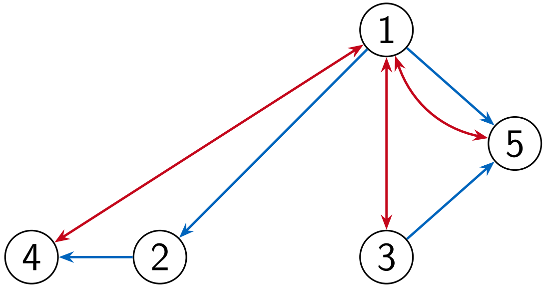
# Marginal Independence Model

Directed mixed graph $G = (V, D, B)$:

- directed $v \to w$: direct effect $\beta_{vw}$,
- bidirected $v \leftrightarrow w$: latent confounding.




Semiparametric model $\mathcal{M}_G$:

- $\varepsilon: \varepsilon_W \perp\!\!\!\perp \varepsilon_{V \setminus (W \cup \mathrm{sib}(W))}$ for every bidirected connected $W$,
- $X_v = \sum_{w \in \mathrm{pa}(v)} \beta_{wv} X_w + \varepsilon_v, \qquad X = \beta^\top X + \varepsilon.$

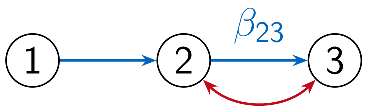
$\varepsilon_2 \perp\!\!\!\perp \varepsilon_{1345}$

$\varepsilon_{24} \perp\!\!\!\perp \varepsilon_{35}$

$\varepsilon_3 \perp\!\!\!\perp \varepsilon_{245}$

$\varepsilon_4 \perp\!\!\!\perp \varepsilon_{235}$

# When Regression Fails




$\varepsilon_2, \varepsilon_3$ dependent

**OLS Estimator.** Regressing $X_3$ on $X_2$ is biased:

$$\frac{\mathrm{E}[X_2X_3]}{\mathrm{E}[X_2^2]} = \frac{\mathrm{E}[X_2(\beta_{23}X_2 + \varepsilon_3)]}{\mathrm{E}[X_2^2]} = \beta_{23} + \frac{\mathrm{E}[X_2\varepsilon_3]}{\mathrm{E}[X_2^2]} = \beta_{23} + \frac{\mathrm{E}[\varepsilon_2\varepsilon_3]}{\mathrm{E}[X_2^2]}.$$

**IV Estimator.** [Wright 1928; Bowden & Turkington 1984]

$$\frac{\mathrm{E}[X_1X_3]}{\mathrm{E}[X_1X_2]} = \frac{\mathrm{E}[X_1(\beta_{23}X_2 + \varepsilon_3)]}{\mathrm{E}[X_1X_2]} = \beta_{23} + \frac{\mathrm{E}[X_1\varepsilon_3]}{\mathrm{E}[X_2^2]} = \beta_{23}.$$

Instrument $X_1$ satisfies

- relevance to target vector: $\mathrm{E}[X_1X_3] \neq 0$.
- orthogonality to target error: $\mathrm{E}[X_1\varepsilon_3] = 0$.

# Identification and Estimation: Half-Trek Criterion (HTC)

For a target $v \in V$, the HTC [Foygel, Draisma & Drton 2012] searches for a **witness set** $Y_v$ with

- $|Y_v| = |\mathrm{pa}(v)|$, $Y_v \cap (\{v\} \cup \mathrm{sib}(v)) = \emptyset$,
- $Y_v$ reaches $\mathrm{pa}(v)$ through half-treks ($\leftrightarrow \cdot \rightarrow \cdots \rightarrow$ or $\rightarrow \cdots \rightarrow$) without sided intersections
- obeying a total **HTC order** $\prec$: $w \in Y_v \cap \mathrm{htr}(v) \Rightarrow w \prec v$.

Each witness $y \in Y_v$ supplies an instrument $Z_y$:

$$Z_y = \begin{cases} X_y, & y \notin \mathrm{htr}(v) \quad \textbf{external}, \\ \varepsilon_y = X_y - \beta_y^\top X_{\mathrm{pa}(y)}, & y \in \mathrm{htr}(v) \quad \textbf{internal}. \end{cases} \quad \Rightarrow \quad \beta_v = \underbrace{\mathrm{E}\left[Z_{Y_v} X_{\mathrm{pa}(v)}^\top\right]^{-1}}_{A_v^{-1}} \underbrace{\mathrm{E}[Z_{Y_v} X_v]}_{b_v}.$$

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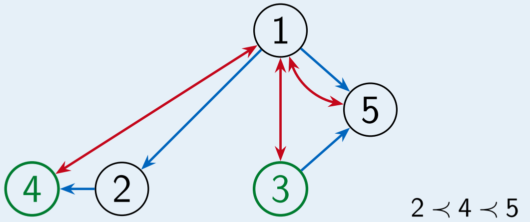
**Example:** Target $v = 5$, $\mathrm{pa}(5) = \{1, 3\}$. $Y_5 = \{3, 4\}$.

- witness 3 **external** ($Z_3 = X_3$),
- witness 4 **internal** ($4 \in \mathrm{htr}(5)$ via $5 \leftrightarrow 1 \rightarrow 2 \rightarrow 4$).

  $\Rightarrow Z_4 = \varepsilon_4$. Requires $\hat{\beta}_{24}$ from an earlier stage.



# The Half-Trek Estimator

- The HTC is a sufficient criterion for identifiability of the parameter matrix $\beta$.
- The HTC identifies the edge parameters $\beta_v$ sequentially through a HTC order $\prec$ and $(Y_v : v \in V)$.
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The empirical HTC estimator

$$\hat{\beta}_v = \hat{A}_v^{-1}\hat{b}_v, \qquad A_v = \mathrm{E}\left[Z_{Y_v} X_{\mathrm{pa}(v)}^\top\right], \qquad b_v = \mathrm{E}[Z_{Y_v} X_v].$$

is a **Z-estimator**: It solves the identifying equation

$$\mathrm{E}[Z_{Y_v}\varepsilon_v] = 0 \quad \Leftrightarrow \quad \mathrm{E}\left[Z_{Y_v}(X_v - \beta_v^\top X_{\mathrm{pa}(v)})\right] = 0. \qquad \text{(IV-orthogonality)}$$

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As a regular estimator it is **asymptotically linear**. Characterized by the **influence function** $\phi_{\beta_v}$:

$$\sqrt{n}(\hat{\beta}_v - \beta_v) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \phi_{\beta_v}(X^{(i)}) + o_P(1) \xrightarrow{d} \mathcal{N}\Big(0,\ \mathrm{E}\Big[\phi_{\beta_v}\phi_{\beta_v}^\top\Big]\Big).$$

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$\downarrow$

**Standard Errors, Tests, CI-Regions**

# Half-Trek Influence Function $\phi_{\beta_v}$

Differentiate row $y$ of $A_v\beta_v = b_v$ (i.e., residual $e_y^\top(b_v - A_v\beta_v)$) along a parametric path for its IF:

$$R_{y,v} = \begin{cases} X_y\,\varepsilon_v, & y \notin \mathrm{htr}(v) \quad \text{external},\\ \varepsilon_y\varepsilon_v - \sum\limits_{q\in\mathrm{pa}(y)} \mathrm{E}[X_q\varepsilon_v]\,\phi_{\beta_{qy}}, & y \in \mathrm{htr}(v) \quad \text{internal}. \end{cases}$$

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**Theorem.** For every HTC-identified directed mixed graph, an IF of the half-trek estimator is

$$\phi_{\beta_v} = A_v^{-1}R_v, \qquad R_v = (R_{y_1,v}, \ldots, R_{y_k,v})^\top$$

The correction term is the price of an estimated instrument due to the finite (!) recursion.

Sanity check classical IV: $R_{1,3} = X_1\varepsilon_3$, $A_3 = \mathrm{E}[X_1X_2]$, recovering $\phi_{\beta_{23}} = X_1\varepsilon_3/\mathrm{E}[X_1X_2]$.

# Asymptotic Inference

**Proposition.** The half-trek estimator is asymptotically normal,

$$\sqrt{n}(\hat{\beta}_{v,n} - \beta_v) \xrightarrow{d} \mathcal{N}(0, \mathcal{V}_v), \qquad \mathcal{V}_v = \mathrm{E}\left[\phi_{\beta_v}\phi_{\beta_v}^\top\right] = A_v^{-1}\,\mathrm{E}\left[R_v R_v^\top\right] A_v^{-\top}.$$

The recursive covariance $\mathrm{E}\left[R_v R_v^\top\right]$ reduces to finitely many fourth-order moments of observables.

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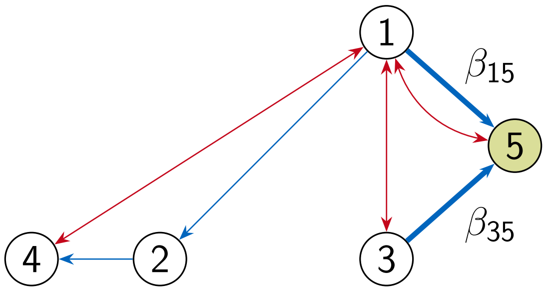
Information presented in `summary(htcfit())`:

- **Standard error / marginal interval:** $\hat{\beta}_{p_j v} \pm z_{\alpha/2}\sqrt{\hat{\mathcal{V}}_v[j,j]/n}, \quad z = \hat{\beta}_{p_j v}/\sqrt{\hat{\mathcal{V}}_v[j,j]/n}$.
- **Confidence ellipsoid:** $n(\hat{\beta}_v - \beta)^\top \hat{\mathcal{V}}_v^{-1}(\hat{\beta}_v - \beta) \leq \chi^2_{k,1-\alpha}$.
- **Wald test** of $H_0: C\beta_v = c$: $W_n = n(C\hat{\beta}_v - c)^\top (C\hat{\mathcal{V}}_v C^\top)^{-1}(C\hat{\beta}_v - c) \xrightarrow{d} \chi^2_r$.

# Calibration

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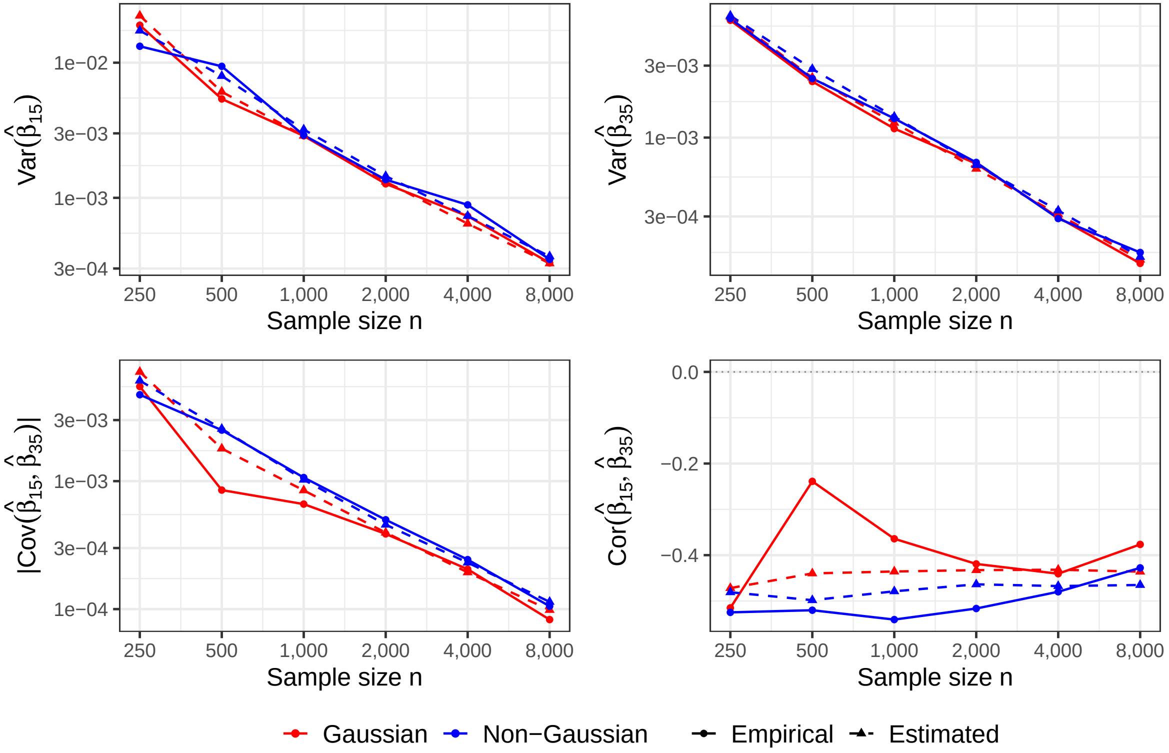
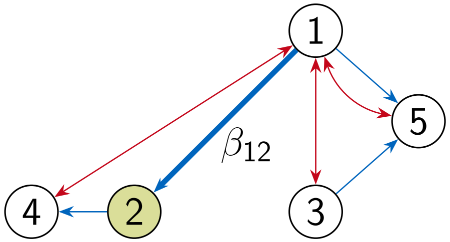
Well calibrated under Gaussian *and* non-Gaussian errors.

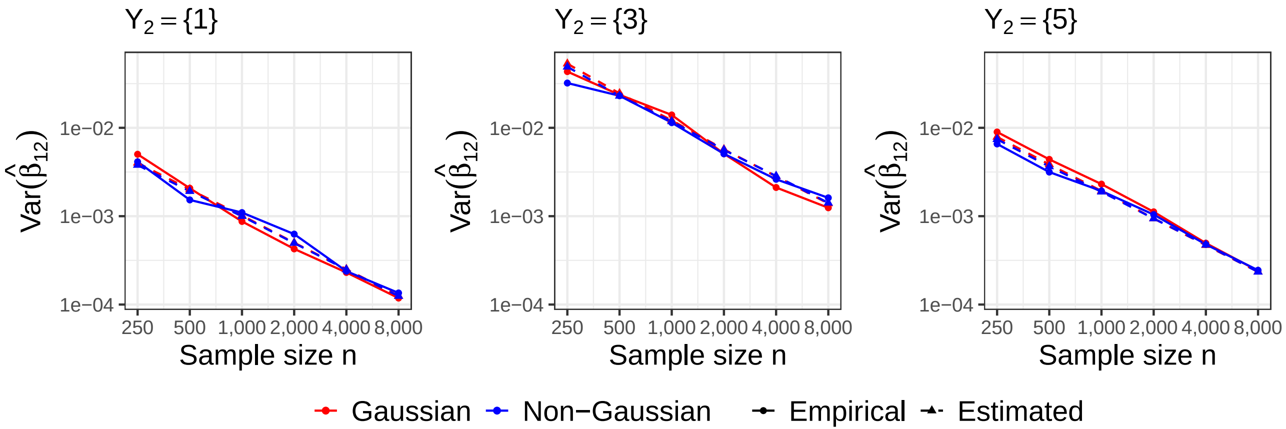


# HTC Witness Choice Matters

Node 2 admits three valid witness sets $Y_2 \in \{\{1\},\{3\},\{5\}\}$. All identify $\beta_{12}$, but with different variance:




$v = 2, Y_2 \in \{\{1\},\{3\},\{5\}\}$



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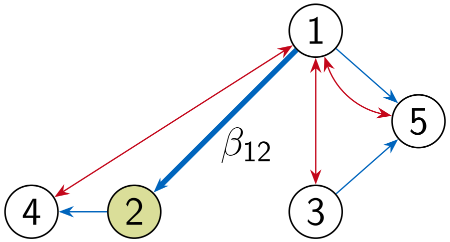


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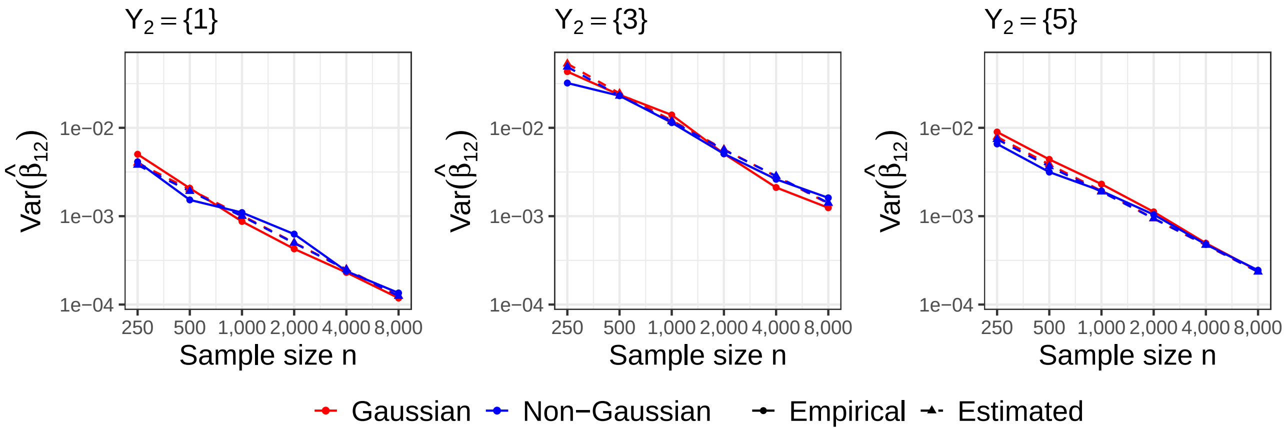
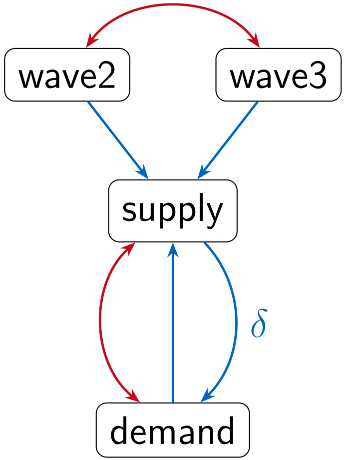



Variance-optimal witness choice *in advance* is an **open problem**.

# Real Data: The Fulton Fish Market [Graddy 1995; Angrist & Graddy 2000]

$n = 97$. Daily whiting transactions.




- Target: $\delta = \beta_{supply,demand}$.
- $\hat{\delta}_{\mathrm{OLS}} = -0.525\ (\pm 0.172)$.
- $\hat{\delta}_{\mathrm{Angrist,2SLS}} = -1.01\ (\pm 0.42)$

```
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supply -> demand     -0.8410      0.3827  -2.1976     0.028 *
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supply -> demand     -0.7611      0.4246  -1.7926     0.073 .
---
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual std. dev: 0.6798   Structural R-sq: 0.0708   (n = 97)
-------------------------------------------------------
```

# Contributions

- **Influence function** $\phi_{\beta_v} = A_v^{-1} R_v$ for every HTC-identified directed mixed graph, cyclic ones included.
- **Closed-form asymptotic variance** $\mathcal{V}_v$ by descent through $\prec$. Distribution-free, no bootstrap.
- **Full inference**: Confidence region, marginal interval, Wald test. Our `summary(htcfit())` table in `R`.

Open: 1) Variance-optimal witness set / ordering in advance 2) Characterizing semiparametric efficiency.


Foygel, Draisma & Drton (2012). *Half-trek criterion for generic identifiability of linear structural equation models.* The Annals of Statistics.

Angrist, Graddy & Imbens (2000). *The Interpretation of Instrumental Variables Estimators in Simultaneous Equations Models with an Application to the Demand for Fish.* The Review of Economic Studies.

Mareis, Sturma & Drton (2026). *Semiparametric Inference for Half-Trek Estimators in Linear Structural Equation Models.* PMLR (to appear).


Website

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